Back to Snowflakes, and beyond

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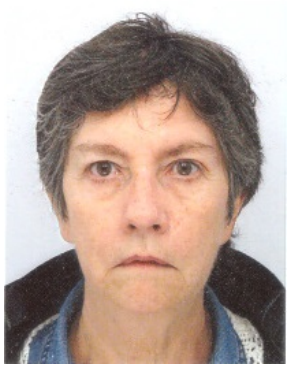

Abstract

I already explored the fascinating world of snowflakes for GA 2016. Being mostly interested in the relation of snowflakes to grammars at that time, I developed a fractal model based upon IFS. But the natural way of thinking of the growth of snowflakes is rather through hexagonal-cell cellular automata. I had a first attempt at that with a hybrid integer-value CA, combining a two-layer CA with the classical one.

Though all these experiments were providing aesthetically interesting results, those were disappointing as realistic snowflakes. In particular, they did not feature the famous “dendrites”, characteristic of a lot of snowflakes.

I recently discovered a 2004 paper by Clifford A. Reiter, in which he proposes a real-value CA, able to generate those hoped for dendrites.

In this paper I expose and implement his model and show how efficient it indeed is. Analysing it raises some questions, for instance thresholds beyond which dendrites emerge.

Beyond that implementation and exploration, this paper extrapolates this model to other tessellations, such as triangular- or square-cell frames.

Lastly, this paper explores the links between real-value averaging neighbourhood models, their capacity to provide emerging patterns, and their interest for generative art.

1. Snowflakes as cellular automata

Snowflakes, or, should we say, snow crystals (because actual snowflakes are generally constituted of a few intertwined snow crystals) have fascinated many people for a long time, for their six-fold symmetry and their great diversity [1]. A

French mathematician, Etienne Ghys, is one of the most recent representatives of this fascination [2]. Though he does not himself provide any new insight about the formation of snow crystals, his book made me discover a reference I unfortunately missed in 2016, though I should not have, for it is written by one the authors of the paper describing a CA model I did exploit [3].

In 2016, I explored some fractal IFS models, because my aim was not so much to get realistic snow crystals. My entry door into snowflakes was through Jules Bourgoin, and I wanted to see what relationships to grammars snowflakes could have. But it is obvious that considering how snow crystals (as well as any crystal) actually grow, cellular automata are the most likely model candidates, and even strictly growing CA, which are the only ones considered here. As a matter of fact, crystals grow by aggregation to a cluster, by contiguity, and most importantly, they do so on a frame that is a tessellation of space, which is a crucial feature of cellular automata.

An amazing feature of snow crystals, contrary to most other crystals, is that they develop, more or less, in a plane. Surely, there are prism-like snow crystals, but even in those, one sees the specific hexagonal symmetry present in the section of the prism. If snow crystals develop in 3D space, that space if not isotropic: there is a plane in which the hexagonal frame is deployed, plus a linear axis perpendicular to that plane. It is then a legitimate approach to simulate the growth of snow crystals with 2D hexagonal-cell cellular automata.

The fundamentals of cellular automata imply first that cells are all of the same size and are contiguous which each other. Frames are then analogous to tessellations, or tilings, of the space we consider. In 2D, as is well known, only three tessellations are possible: by triangles, by squares, or by hexagons. Square cells are the most used because they are the simplest to simulate, being possible to be represented by pixels in a bitmap. Hexagonal cells cannot be represented in that way, and we have to contrive this hexagonal (or triangular, if we consider the centres of cells) tessellation with the orthogonal nature of bitmaps. Incidentally, one can remark that this ‘nature’ is not essential, for instance the cathode-ray tubes used a hexagonal frame.

Another crucial feature of CA is the definition of neighbours. While for an orthogonal CA there may be an ambiguity, between considering only neighbours that share an edge or including those that share a vertex, for hexagonal cells, it is straightforward: each hexagon has six neighbours and only six.

Now, we can also expand the neighbourhood by adding to the first ‘ring’ of neighbours... And then, ambiguity appears for hexagons. We have six neighbours at the peak of the first hexagonal ring, and six others that nestle between them.

The last feature defining a CA is the notion, and numerical nature, of ‘state’. A state may be an integer, the most simplest version being two-states CA, where the two states may be interpreted as ‘alive/dead’, ‘occupied/free’, and so on. But one can also consider more that

two integer states, or even 'real-value' states (knowing that in any case, those 'real' values are actually 'float' values...).

A CA evolves through a discrete time, the state of any cell at time t+1 being determined by its own state and the states of its neighbours at time t, and only by that. In integer-value CA, one takes into account the number of 'occupied' neighbours, while in real-value CA, there is generally an averaging of states of neighbours.

The simplest hexagonal CA is a two-states CA where states may be interpreted as 'non frozen' and 'frozen', whichever integer we choose to use. A cell 'appears' or is frozen depending on the number of its neighbours, no cell 'disappears', or 'melts'. Each rule is embedded in a binary number, putting a 1 or a 0 depending on the result wished for the rank in the number. One can then translate it into a decimal number for facility. Let's call it the 'classical' CA (Fig.).

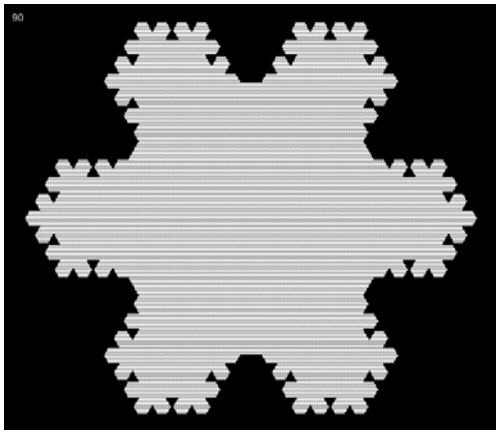

Fig. 1: a result of the 'classical' CA

Results are not bad, they actually meet with a tendency observed in the growth of snow crystals, i. e. growing on the peaks of a previously grown full hexagon. But they obviously lack the complexity of main snow crystals, and certainly not the characteristic dendrites shown for instance in Bentley's photographs [4] (Fig. 2).

Fig. 2: Bentley's microphotograph

2. The quest for dendrites

2.1 A hybridized model

I exposed in 2016 the model elaborated by Coxe and Reiter. It is a somewhat complicated CA, implying a second ring of neighbours and real-value states (for further details see [1]). Beside being complicated, this model does not really provide realistic snowflakes, and yet no dendrites.

Actually, after writing the paper, and being inspired by that model, I found a better way to get more complex and interesting results. First I simplified greatly their model, keeping the idea of a second ring of neighbours, but going back to two integer states. The rules are very close to those of the classical CA, but a condition is added: neighbours are

considered only if they have themselves a neighbour in that same direction.

The result is in itself very disappointing, for it leads to a thin star, but the key is to hybridise it with the classical one, i. e. once in a while activate one of those previous rules (Fig. 3, 4).

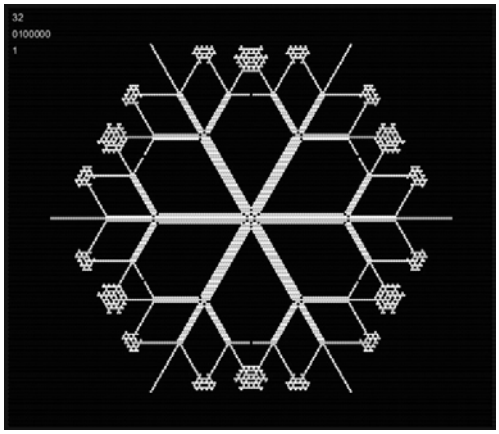

Fig. 3: a result of the hybridized CA

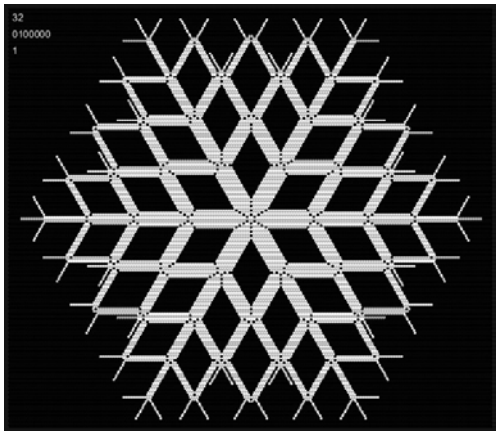

Fig. 4: a result of the hybridized CA

2.2 Reiter's model: description and implementation

Let's now arrive to Reiter's model [*5*] which is the one that incited me to go back to snowflakes.

It is a real-value CA, a cell will be considered as frozen when its value is greater or equal to 1. One affects a value smaller than 1, called β to the whole background, and 1 to one cell (or eventually a cluster of cells) which constitutes the 'seed' of the crystal. At each step, one considers the current state of each cell (for instance such as just described at the start) and initialises the next state at 0 for all cells.

The computing is then performed in two stages. First, one determines cells that are frozen (state >= 0), or neighbours of frozen ones, they are called 'receptive'. The next state of those cells is equal to their current state, with the add of a constant γ. Their current state is put at 0 for the next stage.

The next stage concerns all cells. An averaging is made of the current states of each cell and its neighbours. This averaging is added to the next state of each cell. The averaging gives a weight of 1/2 for the considered cell, and 1/12 for each of its neighbours.

Reiter then observed that, for certain values of β and γ, dendrites finally and happily appear (Fig. 5 where, as in next figures, only states >= 1 are shown, from white (state =1) to grey).

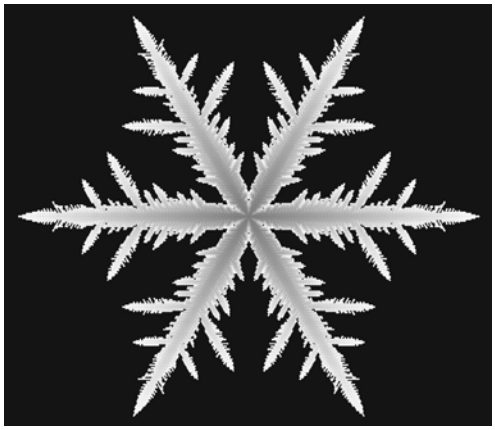

Fig. 5: β=0.4 γ=0.001

Reiter published a table of results for different values of β and γ, in which we see that dendrites appear very specifically for background values around 0.4. These dendrites may be very close from those actually present on snow flakes, while other ones are more fishbone-like or feather-like.

Other results may be called ‘stars’, and other ones ‘patterns’, roughly similar to those obtained for some rules of the classical integer-value CA.

2.3 Reiter’s model: discussion

The additive constant γ is not compulsory in order to get complex (including dendritic) and very diverse results (Fig. 6-12).

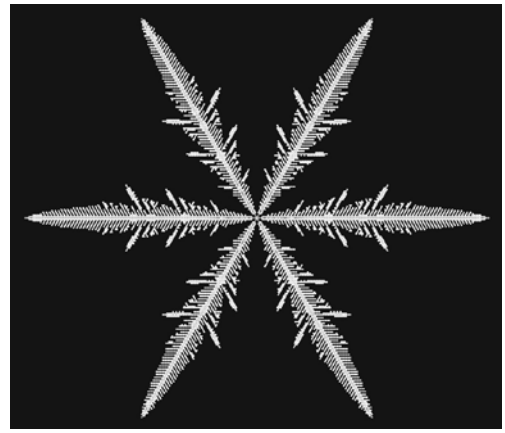

Fig. 6: β=0.3 γ=0

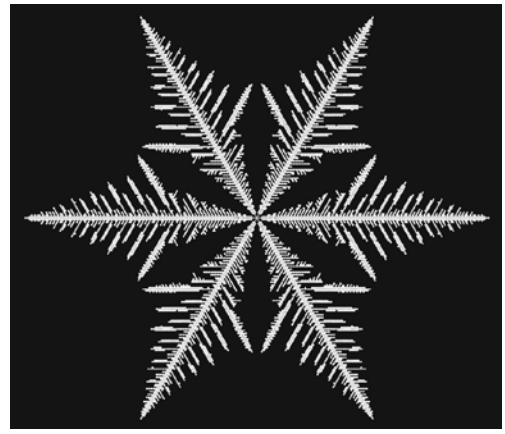

Fig. 7: β=0.4 γ=0

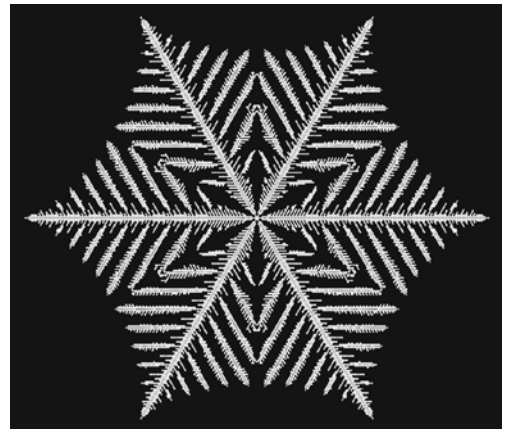

Fig. 8: β=0.5 γ=0

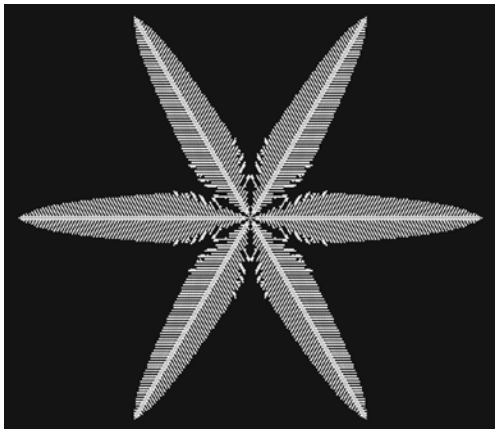

Fig. 9: β=0.6 γ=0

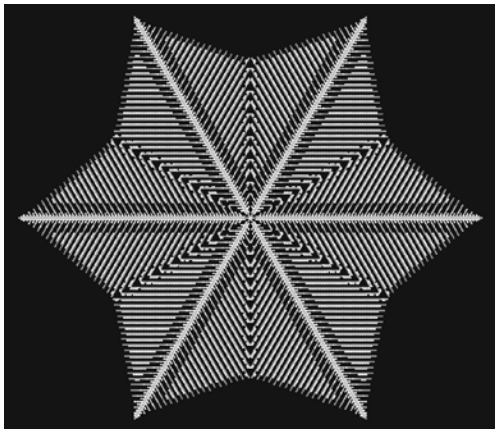

Fig. 10: β=0.7 γ=0

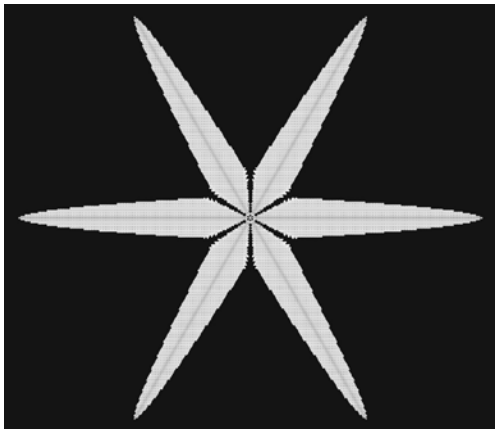

Fig. 11: β=0.9 γ=0

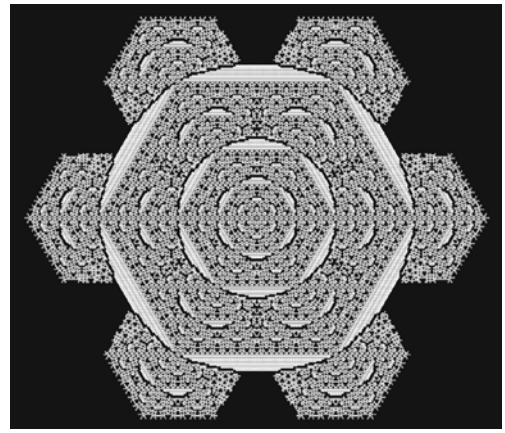

Fig. 12: β=0.99 γ=0

What this sequence shows is that the progression from dendritic to pattern (Fig. 12), passing by petals (Fig. 11), is not a continuous, smooth one. Let us examine what happens for values of β between .9 and .99. There is a sequence of strongly ribbed stars until .929 and an abrupt jump to a typical pattern for .93.

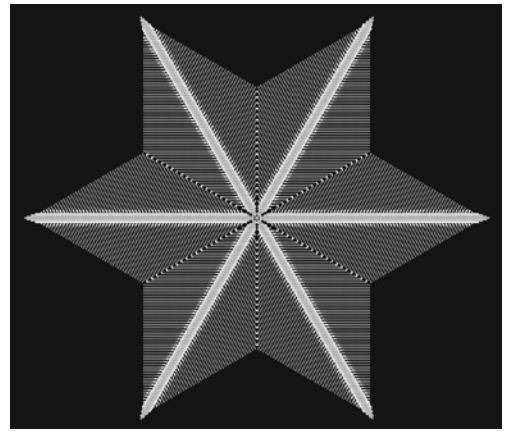

Fig. 13: β=0.929 γ=0

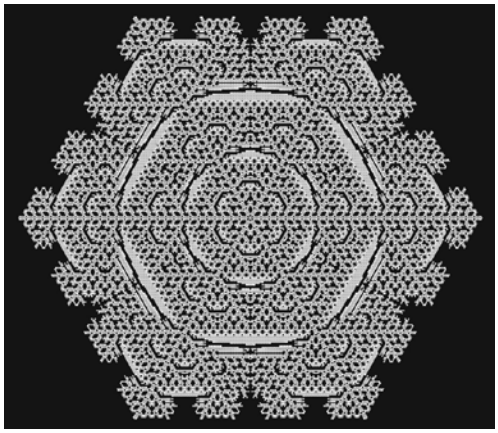

Fig. 14: β=0.93 γ=0

How many decimals we try to add, the model does not seem to yield a smooth transition between those two types of very different results.

Even if we focus on the start of the process, we are not able to catch any transition (Fig. 15, 16 where states < 1 are shown in values of blue).

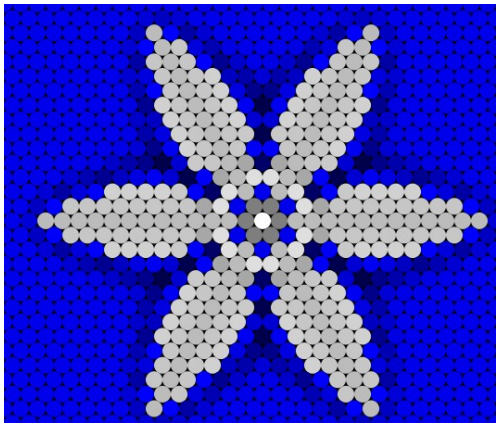

Fig. 15: β=0.9295 γ=0

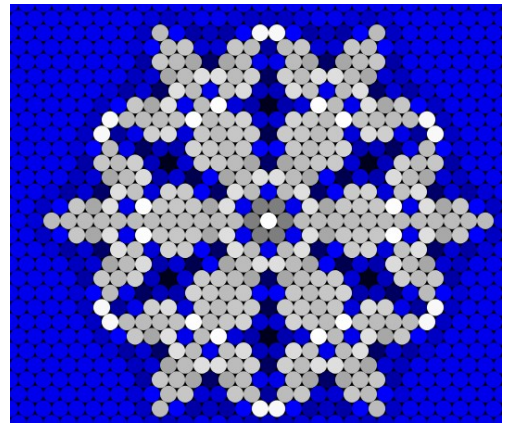

Fig. 16: β=0.92958 γ=0

Reiter's arguments justifying his use of a real-value CA, the meaning of values of background and additive constant, and the averaging process, are convincing. It is plausible that some process of diffusion (mimicked by the averaging) takes place, with changes of temperature leading to ice as soon as it attains 0°C.

Is Reiter's model the ultimate solution to the snowflake problem? It indeed catches some features such as dendrites that are not attained by other models. But some other configurations are not obtained, while some results of the model do not seem to correspond to any actual snowflake.

This model has been pushed further towards an even more accurate correspondence to actual snowflakes by Gravner and Griffeath (applets have been developed for instance here [6]) who define and use seven parameters instead of two. While results are indeed very interesting, one may regret the simplicity of Reiter's which can, even with only one parameter, produce a great variety of complex results.

All the more because this unique parameter or those two, may be tampered with during the process, mimicking the fact that environmental conditions (such as the temperature and hydrometry) vary during the growth of the snow crystal.

3 Extrapolations

Going beyond snowflakes one can explore other tessellations of the plane.

It is well know that the plane is tiled either by hexagons, triangles, or squares. Snowflakes are well simulated by hexagonal-cell CA because of the structure of the water molecule, but we can imagine, even if they don't exist in nature, molecules that would have triangular or square structures, and CA that would give three or four-fold symmetrical results.

Triangular-cell CA are actually more complex than hexagonal-cell ones, because, first, the tiling triangles present themselves in two directions (while all hexagons are identical by translation) and secondly, because we may consider different neighbourhoods: either only the triangles that are adjacent, or also those that touch the peaks of the first triangle.

Even taking the simplest 3-neighbours neighbourhood, triangular-cell CA are disappointing, in the sense that they don't give three-fold symmetry. As a matter of fact, those three first neighbours have themselves two neighbours, and that leads to six branches, not three... Results are then comparable though not absolutely identical) to hexagonal-cell ones (Fig. 17, 18).

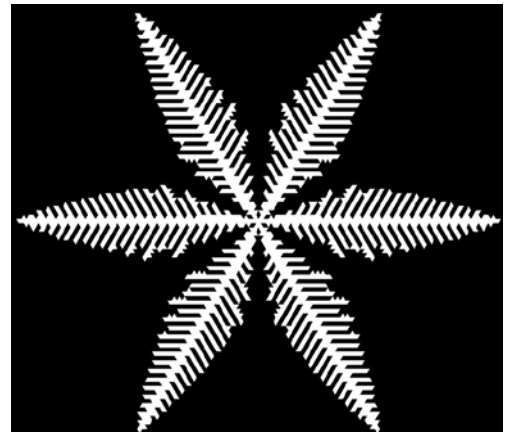

Fig. 17: β=0.5 γ=0

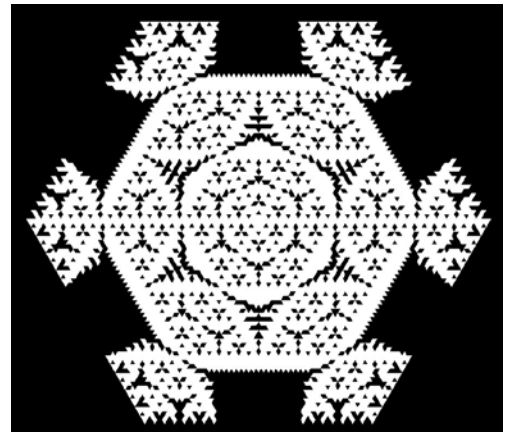

Fig. 18: β=0.95 γ=0

Square-cell CA are more promising. We can consider two types of neighbourhood: four (von Neumann) or eight (Moore) neighbours.

A CA model similar to Reiter's one but for an orthogonal grid has been thoroughly examined by Zhao *et alii* [*7*].

Zhao favours the Moore neighbourhood, but the von Neumann one is not to be neglected. One can see dendrites appear as in the hexagonal model, but patterns encountered for higher values of β are rather different (Fig. 19, 20).

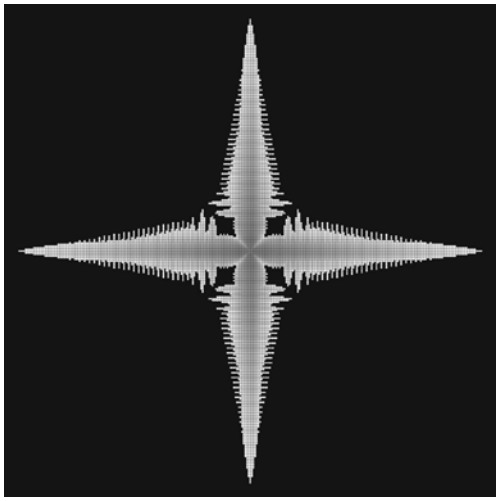

Fig. 19: β=0.4 γ=0.003

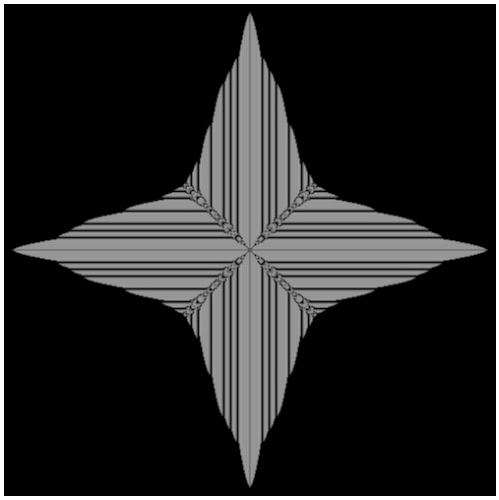

Fig. 20: β=0.99 γ=0

One advantage of the orthogonal grid is that we can play with the borders, by prolonging the process in the square equipped with a toric topology (Fig. 21), possibly with a different resolution (Fig. 22).

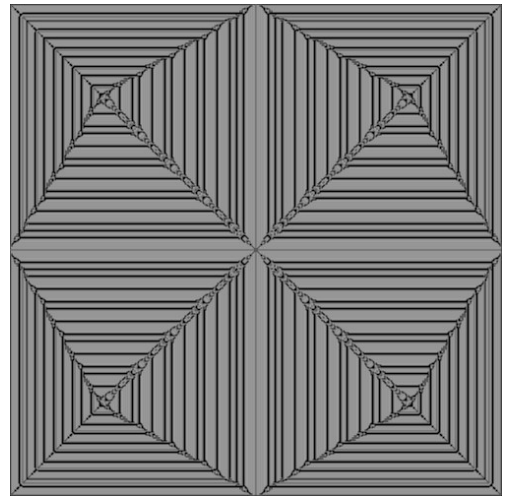

Fig. 21: β=0.99 γ=0

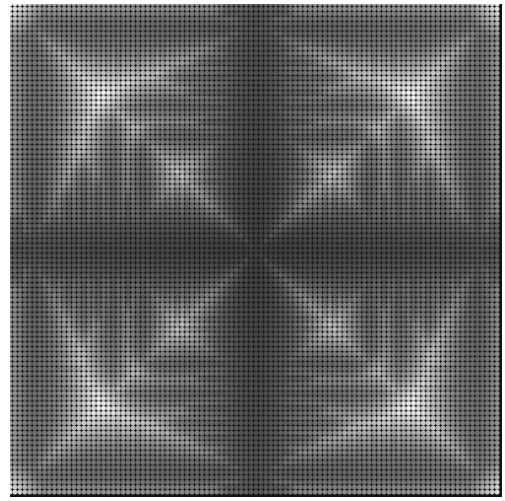

Fig. 22: β=0.4 γ=0.003

The Moore neighbourhood presents a problem which we have not encontered yet. A crucial feature of all real-value CA is the averaging of the values of the neighbourhood. In the cases of orthogonal, triangular and square von Neumann cells, all neighbours are adjacent, they share an edge with the considered cell, so their distance is the same, and the averaging is

straightforward: the sum of the values divided by the number of neighbours. But with the von Neumann neighbourhood, four neighbours are at the corners, so that their distance is larger than that of the four other neighbour. Zhao proposes a formula with a weighted averaging respecting this difference.

However, this caution is not mandatory. Taking the sum of values and dividing by eight works just the same as the weighted averaging.

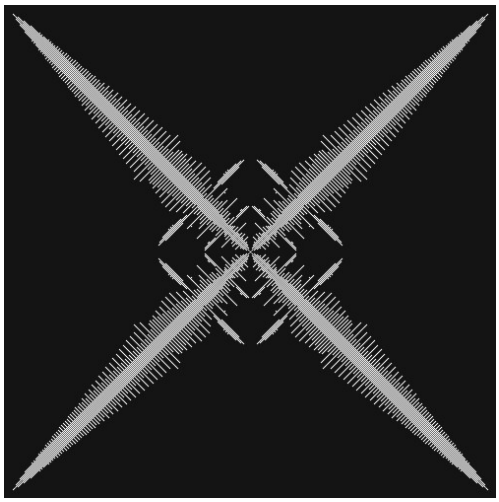

Fig. 23: β=0.25 γ=0

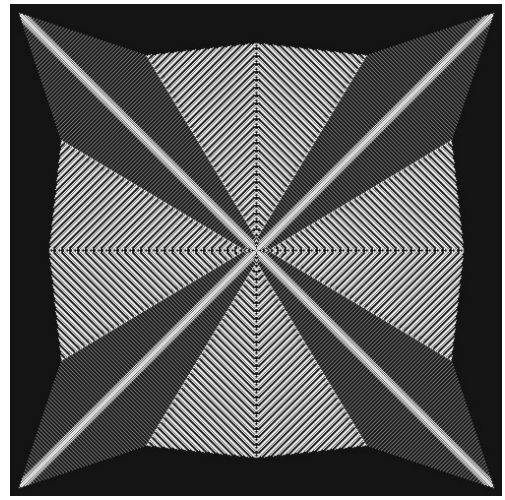

Fig. 24: β=0.9 γ=0

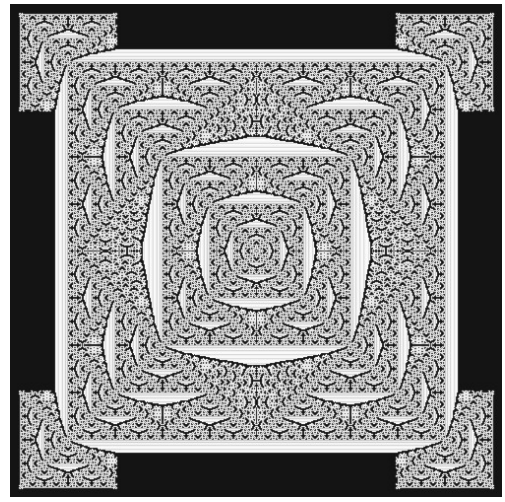

Fig. 25: β=0.99 γ=0

We encouter dendrites (Fig. 23), and patterns very similar to the hexagonal ones (Fig. 25), with a lot of other patterns where, in a way, dendrites join themselves (Fig. 24).

Zhao affirms seeing secondary and even tertiary dendrites in some results, though it is not so clear, either looking at his pictures or ours. However, dendritic results show some resemblance with

pictures of actual crystals, namely those of ammonium bromate NH_4Br. One could pursue this exploration with a 3D CA, of course.

In conclusion, beyond the quest for snowflakes dendrites, real-value CA prove to be a valuable asset for generating emerging patterns. The notion of diffusion they simulate makes them close, though simpler, to reaction-diffusion models which we know provide so many amazing patterns.

References

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