

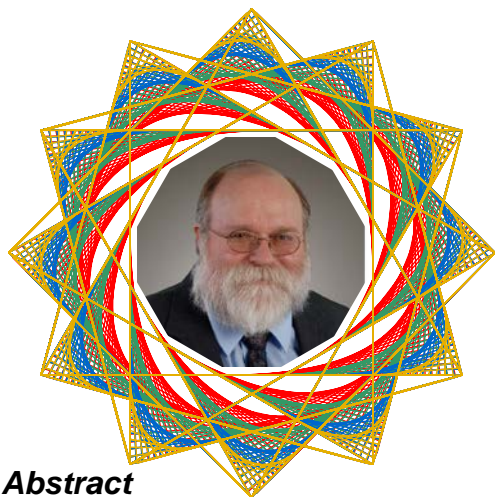
# Electronic String Art as Educational Algorithmic Art

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## Abstract

Traditional string art creates curves from straight lines using nails, string, and a simple counting rule such as connect every 7<sup>th</sup> nail. Different counting rules or nail locations produce new images, but each change requires significant setup time. By contrast, *electronic string art*, ESA, images change instantaneously as parameters change. This allows users to rapidly adjust values and quickly test hypotheses even if those hypotheses are only incompletely understood. Such purposeful play can lead to learning, even if that learning is informal in nature.

## 1. The Four ESA Parameters

Electronic String Art is a suite of models, each based on a closed set of connected line segments using a simple counting

rule [1]. ESA has two image creation modalities, web and Excel, that complement one another [2].

ESA images are framed on regular  $n$ -gons or  $n, J$ -star.  $n$  and  $J$  determine the *vertex frame*, VF, shown in blue in Fig 1. Each line of the VF is subdivided into  $S$  equal subdivisions creating at most  $n \cdot S$  possible subdivision endpoints (think nail locations), shown as red dots in Fig 1. The image is created by drawing a line connecting every  $P^{\text{th}}$  endpoint starting at the top. This process is repeated until the top is once again achieved in either  $n \cdot S$  or fewer lines. The bottom row shows two 15-line ESA images using this algorithm with first line overlaid in red.

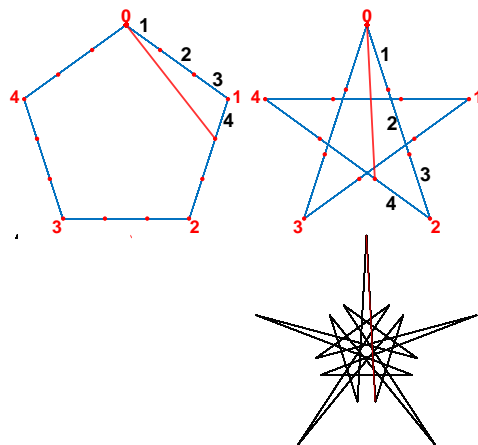


Fig 1. Examples showing ESA images created given  $n = 5$ ,  $S = 3$ , and  $P = 4$  for  $J = 1$  left and  $J = 2$  right. Top row shows VF in blue with 3 equally spaced

subdivisions per VF line and first line drawn (shown in red) based on counting rule that states: Draw a line connecting every 4<sup>th</sup> red subdivision endpoint. The bottom row shows the resulting images.

This simple structure allows wide latitude to explore ideas even if those ideas are incompletely understood. Users can rapidly test hypotheses and search for similarity simply by playing with the model. In the process, users are able to visually understand mathematics more deeply, even if that understanding is simply a by-product of such play. This purposeful play is consistent with Stein's *Triex: Explore, Extract, Explain* view of how to teach mathematics [3].

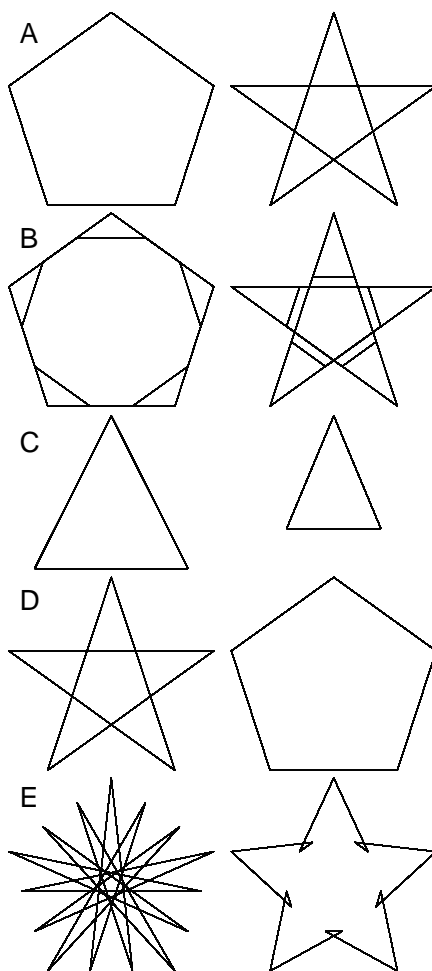
Images change as underlying parameters change and those changes are based on mathematical rules. But there is a second form of dynamism in ESA. It involves how a specific image is drawn, line by line.

## 2. Issues of Commonality

The completed image connects start of the first line with the end of the last line using a simple counting rule. As such, there are two ways to draw each static image. Additionally, the number of lines required, and how subsets of those lines relate to one another depend on commonality between the parameters.

### 2.1 Distinct Images

An ESA image has one line in and one line out of the top of the image. The values of  $n$ ,  $S$ ,  $P$ , and  $J$  determine how this first line (shown in red in Fig 1), and hence the image, is drawn. Therefore, there are two ways to draw any image. This is examined in Fig 2 which shows all distinct images given  $n = 5$  and  $S = 3$  for  $1 \leq P < 15 = n \cdot S$ . Between Figs 1 and 2, there are 10 distinct ESA images.



$J = 1, 4$  Left\*

| Fig 2 Row | $J = 2, 3$ Right | $15 - P$ | Lines in Cycle | Total |
|-----------|------------------|----------|----------------|-------|
| A         | 1                | 14       | 3              | 15    |
| B         | 2                | 13       | 3              | 15    |
| A         | 3                | 12       | 1              | 5     |
| C         | 5                | 10       | 3              | 3     |
| D         | 6                | 9        | 1              | 5     |
| E         | 7                | 8        | 3              | 15    |
| Fig 1     | 4                | 11       | 3              | 15    |

\*Notes:  $J$  and  $5 - J$  given  $n = 5$ .

$P$  and  $15 - P$  given  $nS = 15$ .

$(J_0, P_0)$  and  $(n - J_0, nS - P_0)$  are drawn in one direction and  $(J_0, nS - P_0)$  and  $(n - J_0, P_0)$  are drawn in the other.

Fig 2. Distinct ESA images:  $n = 5, S = 3$ .

We see that one may restrict  $J < n/2$  and  $P < n \cdot S/2$  because larger values of  $J$  and  $P$  only alters the direction that the image is drawn. We also see that if  $P$  is a factor of  $S$ , then the image is simply the VF although note that if  $P = 1$  or  $14$  the VF is drawn in 15 connected segments, but if  $P = 3$  or  $12$ , the VF is drawn in 5.

## 2.2 Vertex Common Factor

The first form of commonality is in the VF between the number of vertices,  $n$ , and the number of jumps between vertices,  $J$ . This is called the *vertex common factor*,  $VCF = \text{GCD}(n, J)$  where GCD is the greatest common divisor function. The number of used vertices and lines in the VF is  $n/VCF$ .

The simplest example of  $VCF > 1$  is a 6,2-star whose VF is an equilateral triangle because the circuit is complete once the even vertices are connected with one another. A 6,2-star with 6-points cannot be continuously drawn.  $VCF > 1$  is also the reason that a 12,5-star is the only distinct ESA star with 12-points (as noted above, a 12,7-star simply draws the VF in the reverse order). In general, as mentioned in the Fig 2 notes, an  $n, J$ -star and an  $n, (n-J)$ -star simply differ in the direction in which the star is drawn.

## 2.3 Subdivision Common Factor

Given  $n/VCF$  used vertices in the VF and  $S$  subdivisions per VF line, there are  $S \cdot n/VCF$  possible subdivision endpoints that can be used in the image. The second form of commonality is the *subdivision common factor*, SCF, where  $\text{SCF} = \text{GCD}(P, S \cdot n/VCF)$ . The number of used endpoints (and lines) in the image is  $S \cdot n/VCF/\text{SCF}$ . In Fig 2 Row C, the images are both isosceles triangles connecting subdivision endpoints 5, 10,

and 0 since  $\text{SCF} = 5$ .  $P = 3$  creates Row A connecting subdivision endpoints 3, 6, 9, 12, and 0 and the image is the VF but  $P = 6$  connects endpoints 6, 12, 3, 9, and 0; a pentagram results if  $J = 1$ , and a pentagon results if  $J = 2$ . In each case, this happens because  $\text{SCF} = 3$ .

## 2.4 Cycles (Cycle Generator)

The image will have rotational symmetry equal to the number of polygonal vertices included in the final image. Given this, in order to understand an image, one need only consider that part of the image spanning from the top of the image (the start of the first line) to the next time a polygonal vertex is encountered using the counting rule described in Section 1. For simplicity we call such a sub-image a *cycle*. [It is worth noting at the onset that it is mathematically more accurate to call this a *cycle generator* because the entire image is a cycle.] In Fig 1 the first cycle ends at vertex **4** for the left image and vertex **3** for the right image if  $P = 4$ . The next cycle ends at vertex **3** at left and **1** at right, and so on. As noted in Fig 2, the same static image occurs given  $P = 11$  in which case the first cycle ends at vertex **1** for the left and vertex **2** for the right.

In general, the number of lines in a cycle is  $S/\text{GCD}(S, P)$ . As a result, given  $S = 3$  in Fig 2,  $P = 3, 6, 9$ , and  $12$  produce one-line cycles; all other  $P$  have 3-line cycles.

## 3. Searching for Similarity

From an educational perspective, the best question you can ask is: *Can you find similar but not identical images to an image you find interesting?*

Suppose, for example, that you like the spiky 5-needle image in Fig 1,  $(n, S, P, J)$  of  $(5, 3, 4, 2)$ . You might quickly find similar images such as  $(7, 3, 4, 3)$ , or  $(7, 3, 5, 2)$ , or

(11,3,4,5). In each case this involved changing just 2 of the 3 parameters  $n$ ,  $P$ , and  $J$  from their Fig 1 values while maintaining  $S = 3$ . If one allows larger  $S$  values then the images become more complex, nonetheless, some nice spiky images are still possible as Fig 3 shows. *The key is to find a subdivision endpoint that is very close to the vertical diameter.*

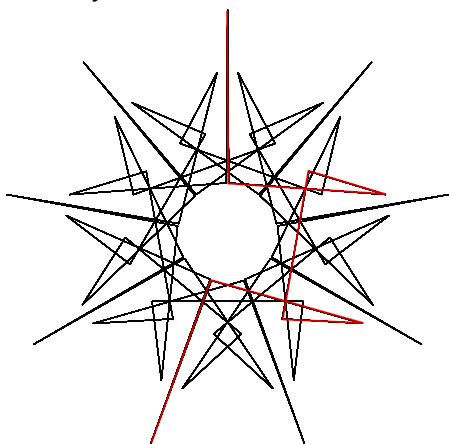


Fig 3. Image given  $(n,S,P,J) = (9,7,17,4)$  with 7-line first cycle overlaid in red.

In Fig 3, the 3<sup>rd</sup> subdivision endpoint from each end of every line of the VF is very close to another endpoint on the VF. The pair that are very close to the vertical diameter are  $P = 17$  and  $P = nS - 17 = 46$ . The version shown is the 3<sup>rd</sup> point of the 3<sup>rd</sup> VF line (since  $17 = 2 \cdot 7 + 3$ ), just to the right of the centerline. The end of first cycle is at vertex 5. The other drawn version,  $P = 46$  is the 4<sup>th</sup> point on the 7<sup>th</sup> VF line (since  $46 = 6 \cdot 7 + 4$ ), just to the left of the centerline and the first cycle in this case would end at vertex 4.

#### 4. Some Interesting Images

Playing with these parameters produces a wide range of images. Here are a few examples. Fig 4 is a 138-line spinning needle star that uses all its vertices but

only 1/5<sup>th</sup> of its 690 subdivision endpoints since  $SCF = 5$ . These endpoints and the first line are shown in red. Given this first line, it is clear the image is drawn counterclockwise,  $\cup$ . If you want to see it drawn clockwise,  $\cup$ , change to  $J = 17$ .

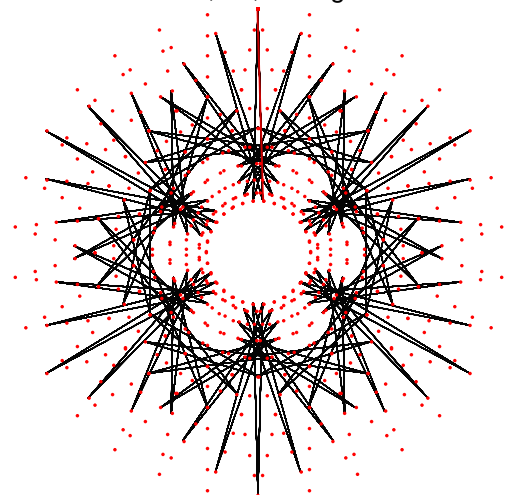


Fig 4. Spinning Needle Star image given  $(n,S,P,J) = (30,23,265,13)$ .

Figs 5 and 6 show images with first cycle overlays in red. They differ by a single subdivision per VF line ( $S = 11$  vs. 12).

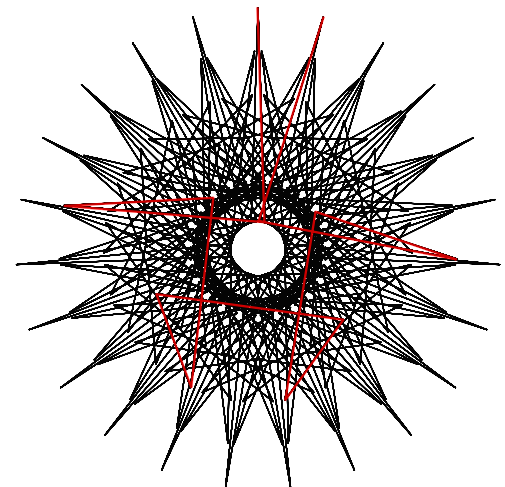


Fig 5. Needle Star with Shields image given  $(n,S,P,J) = (23,11,71,12)$ .

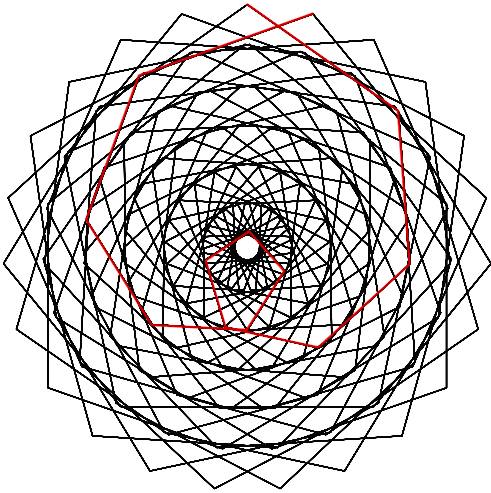


Fig 6. Stacked Circles image given  $(n,S,P,J) = (23,12,71,12)$ .

Because the first cycle ends at vertex 1 in Figs 5 and 6, both are drawn in a  $\cup$  1-time-around fashion. The same would be true for other values of  $S$  because  $P$  and  $J$  are modular multiplicative inverses (MMI) modulo  $n$  meaning that  $71 \cdot 12 = k \cdot 23 + 1$  for some  $k$  (here  $k = 37$ ). This is like the *Ticking Clock* images in [4].

The 570 line Fig 7 provides a very different visual interpretation of MMI. Here 7 and  $P$  are MMI mod  $n \cdot S$  ( $7 \cdot 163 = 1141 = 2 \cdot 570 + 1$ ) meaning that the end of the 7<sup>th</sup> image line is the 1<sup>st</sup> subdivision on the 1<sup>st</sup> VF line, just beneath and to the right of the top of the image at the end of the 7<sup>th</sup> red line. I call such sub-images *single-step*. Given this, the 14<sup>th</sup> endpoint is the 2<sup>nd</sup> subdivision endpoint, and so on. The image is completed after about 81 steps (since  $81 = 567/7$ ). Note that  $J$  is not part of this MMI calculation so that changing  $J$  creates additional single-step images (try  $J = 7$ ). These images are best viewed dynamically using links in the **Notes** [2].

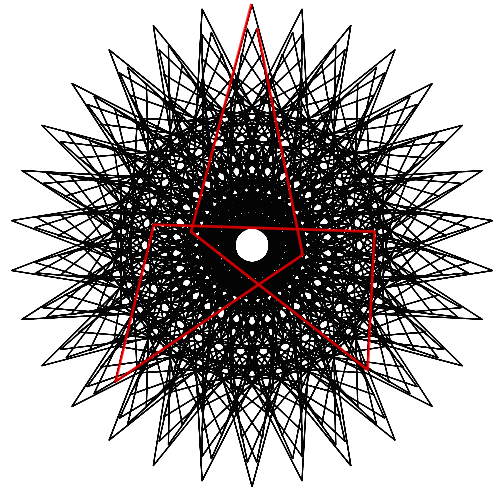


Fig 7. Three Shape-Shifting Triangles image given  $(n,S,P,J) = (30,19,163,13)$ .

Fig 8 is a divisible star. This 250-line image uses every vertex,  $VCF = 1$ , but only one 248<sup>th</sup> of possible subdivisions,  $SCF = 248$ , because  $n = P$ . The first 3 lines, shown in orange form an “almost-equilateral triangle” the peak of which rotates  $\cup$ , and is a *smallest-step* [1].

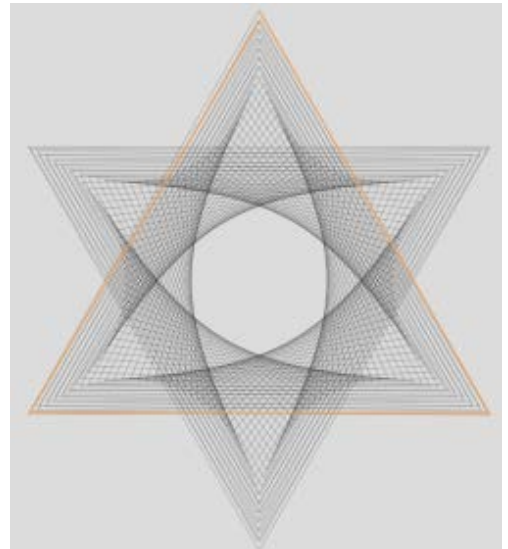


Fig 8. An example of the illusive 6,2-Star given  $(n,S,P,J) = (248,250,248,83)$ .

The entire image is created by 83 of these almost equilateral triangles rotating  $\cup$  plus 1 line ( $250 = 3 \cdot 83 + 1$ ) which is why the last triangle ends at "vertex 4" of the "6,2-star" so that the final line brings us back to the top and completes the circuit. This description is best understood using the web link to watch it get drawn dynamically. To see a virtually identical image get drawn  $\cup$  in 248 lines, change to (250,248,250,83), since  $248 = 3 \cdot 83 - 1$ .

## 5. Jump Sets

Jump sets are an extension of the basic model that creates a more complex VF. The easiest way to explain jump sets is to set  $n = 12$  since then we can talk in terms of hour jumps.  $J = 2$  produces a hexagon connecting 2-4-6-8-10-0 (read - as connect) since  $VCF = 2$ . A jump of 4 produces a triangle connecting 4-8-0 since  $VCF = 4$ . But what if a jump of 2 is followed by a jump of 4?

### 5.1 VCF with Jump Sets

The VF of the two-jump set,  $J_1 = 2, J_2 = 4$  is 2-6-8-0. If  $J_1 = 4, J_2 = 2$  the VF is 4-6-10-0. If  $J_1 = 5, J_2 = 1$  the VF is 5-6-11-0. All three VFs are quadrangles (in fact all are rectangles) since the circuit is completed once the top is achieved *at the end of a set*. If  $J_1 = 6, J_2 = 3$  the VF is 6-9-3-6-0-3-9-0. The 8-line 4-leaf clover VF uses only 4 vertices because each used vertex is used twice. If you draw the VF, you will note that both diameter lines are drawn twice, once from one end, and once from the other.

Consider  $J_1 = 3, J_2 = 4$ . Individually, these produce a square and a triangle, but together their  $VCF = 1$ . The first few lines of the VF are 3-7-10-2 ... . The VF has 24 lines with all vertices used twice. Fig 9 shows a four-color example of this image

based on Chapter 17 of [1]. If you look carefully, you can see that the VF has 3 squares and 4 equilateral triangles.

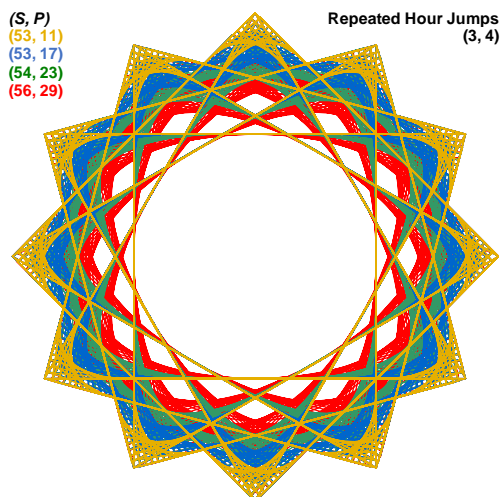


Fig 9. Image based on a double jump set.

This same idea works with different  $n$  and a larger number of jumps in the jump set  $k > 2$ . The general rule for lines in the VF in this instance is  $k \cdot n / VCF$  with  $VCF = GCD(J, k \cdot n)$  and set sum  $J = J_1 + J_2 + \dots + J_k$ .

### 5.2 Zero Jumps

The swirling wreath around the picture on the first page is based on the three-jump set (3,4,0). Compare that image (minus the picture, of course) with Fig 9. The end of the second jump is at 7 in both cases. In Fig 9, the next jump is to 10 which is why you can see a curve spanning vertices 3-7-10 that is absent in the three-jump version. The third jump remains at vertex 7. This is most clearly seen by the red and white swirl and by the white triangular areas like the one bounded by the line from 5 to 8 and the  $15^\circ$  angle, 10-7-11. This  $15^\circ-75^\circ-105^\circ$  triangle is formed by  $J_3 = 0$ . The second set of curves in the swirl is from 7-10-2 but that is the third set of curves in Fig 9.



Suppose we say that the swirl image on the first page is  $\cup$  going toward the center. If you want to change to a  $\cup$  swirl going to the center then simply change to a three-jump set of (4,3,0). These swirls point to the fact that jump set images need not have the vertical symmetry that exists for single jump ESA images. It is worth noting that the four-color models are not available on the web version and a zero jump is attained in the web version by setting  $J = n [1,2]$ .

### 5.3. Fibonacci

Larger jump sets are only available using the web version [2] and are discussed in Chapter 18 of [1]. Among these, many images using Fibonacci numbers as jump sets are beautiful as static images, but are particularly compelling to watch get drawn for those that are single-step.

The first two Fibonacci-based images have a  $k = 7$  jump set using the first 7 Fibonacci numbers, 1,1,2,3,5,8,13 so the set sum is  $J = 33$ . Each is also single-step of length 3 and each shows the first 3 lines in orange.

560-line Fig 10 sets  $n = 132 = 4 \cdot 33$  and  $S = 20$ . The VF has  $28 = 7 \cdot 4$  lines and  $560 = 28 \cdot 20$  subdivision endpoints.  $P = 187$  because  $3 \cdot 187 = 561$  (therefore 3 and 187 are MMI mod 560).

2401-line Fig 11 sets  $n = 231 = 7 \cdot 33$  and  $S = 49$ . The VF has  $49 = 7 \cdot 7$  lines and  $2401 = 49 \cdot 49$  subdivision endpoints.  $P = 1601$  ( $3 \cdot 1601 = 4803 = 2 \cdot 2401 + 1$  so that 3 and 1601 are MMI mod 2401).

Other single-step images are, of course possible. For example, if we start from Fig 11 and change to  $P = 1334$ , a 7,3-star is formed that is single-step of length 9 ( $9 \cdot 1334 = 12,005 = 5 \cdot 1601 + 1$ ). Other MMI pairs are found by backtracking

Euclid's Algorithm as discussed in Chapter 24 of [1].

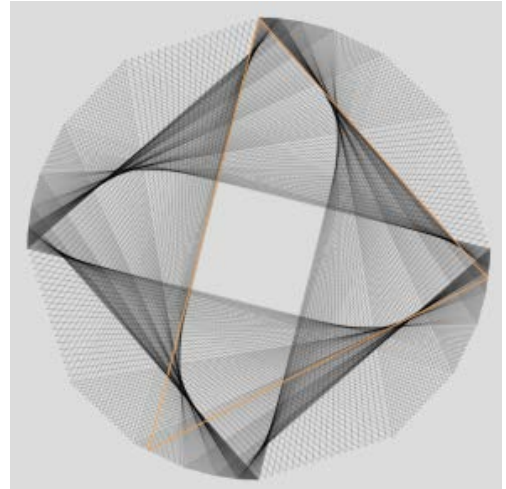


Fig 10. A Fibonacci-7 Cross.

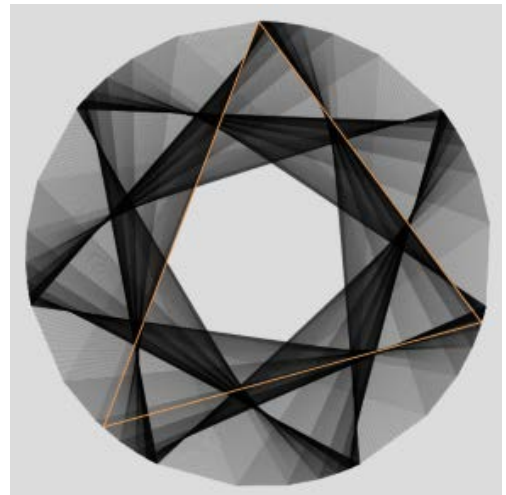


Fig 11. A Fibonacci-7 7,2-gram.

The values used to create Fig 11 were chosen to highlight a way that you can create "flickering videos" from images such as these. Note that  $2401 = 7^4$ . This means that every 7<sup>th</sup>  $P$  value has lower line density because  $SCF = 7, 7^2, \text{ or } 7^3$  and all other  $P$  have  $SCF = 1$ . Put another way, you can watch the image

change in waves by scrolling across  $P$  values and 6 out of 7 will be at full density. How many distinct images are there? Set  $P = 1,200$  and change to 1,201. Both are the same static image; the only difference is the direction in which the image is drawn (see notes to Fig 2). ( $P = 1200$  is a porcupine image [1] as is the  $J = 1$  Row E image in Fig 2.) There are 1,199 distinct images in this situation (since 1 and 7 both produce the vertex frame as discussed in Fig 2).

The bird-beak in Fig 12 is created by setting  $n = 13$ , the size of the 7<sup>th</sup> Fibonacci number. The beak is created with the last half of the set listed backward (with last jump removed) 1,1,2,3,5,8,13,13,8,5,3,2,1 so that  $J = 65$  in 13 jumps. Since  $J$  is a multiple of  $n$ , the image is completed in a single jump set. This image is also single-step of length 2 because  $S = 13$  and  $P = 85$ .

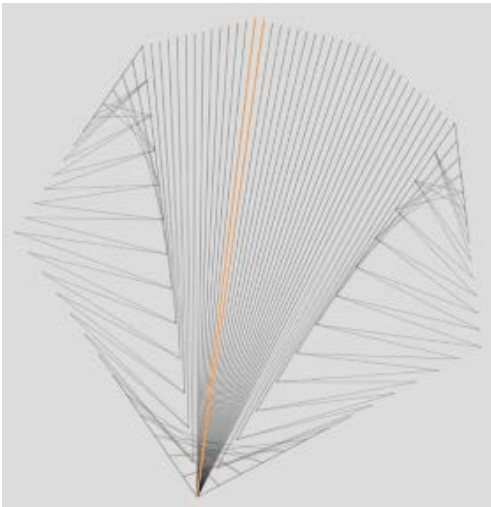


Fig 12. Double Fibonacci Bird-beak.

It is instructive to work through the jumps involved in Fig 11. The point of the beak at vertex 7 is at the end of the 4<sup>th</sup> jump. It is also the location of the pair of 13 jumps

(zero jumps). The bird-beak is another example of a porcupine image.

## 6. Non-Polygonal Vertices

One benefit of working with regular polygons is that users need not worry about where the vertices are located in the  $(x, y)$  plane. Users can explore as soon as they can point and click, and they can watch images change as  $n$  or  $J$  change.

Users can create ESA images with vertices of their own choosing, but the mathematical bar is higher since the user must place those vertices in the  $(x, y)$  plane. This is based on [5] which is expanded on in Chapter 19 of [1].

### 6.1 How $V$ replaces $n$ and $J$

$S$  and  $P$  work just as in the polygonal model but a new variable  $V$  (for *Vertices used*) replaces  $n$  and  $J$ . The key to this version of ESA is that the vertices must form a closed loop. With the vertices of a regular polygon this is true but went unsaid; here it must be explicitly noted in order to understand how the image emerges from the vertex frame. The easiest way to explain how this works is to show how an image like the  $V = 20$  cube in Fig 13 is created from the dashboard shown in Table 1.

After the last used vertex, the next vertex is always vertex 1, here the point  $(10, 0)$ . Given this, a triangle results if  $V = 3$ . When  $V = 5$ , the comma could be the start of the right side or the top. By  $V = 8$  we see that the top is complete and by  $V = 12$  the right side is complete as well. One might think that by  $V = 16$  the image is complete but that does not happen until  $V = 20$ .



As you examine the progression of Fig 13 images, you may wonder why  $V = 9$  looks less complete than  $V = 8$ . The reason is straightforward – vertex 9 is the same as vertex 1; the result is a zero jump like discussed in section 5.2 and also at vertex 7 in the Fig 12 bird-beak.

**20                      19                      11**

|  |                                       |   |
|--|---------------------------------------|---|
| <b>V</b> , # of vertices used. $V \leq 21$ | <b>S</b> , # of subdivisions per side | <b>P</b> , # of subdivisions between Points |
|--|---------------------------------------|---|

**You can change the yellow cells**

*Note* : Keep  $V * S < 1000$  (or pattern will not repeat)

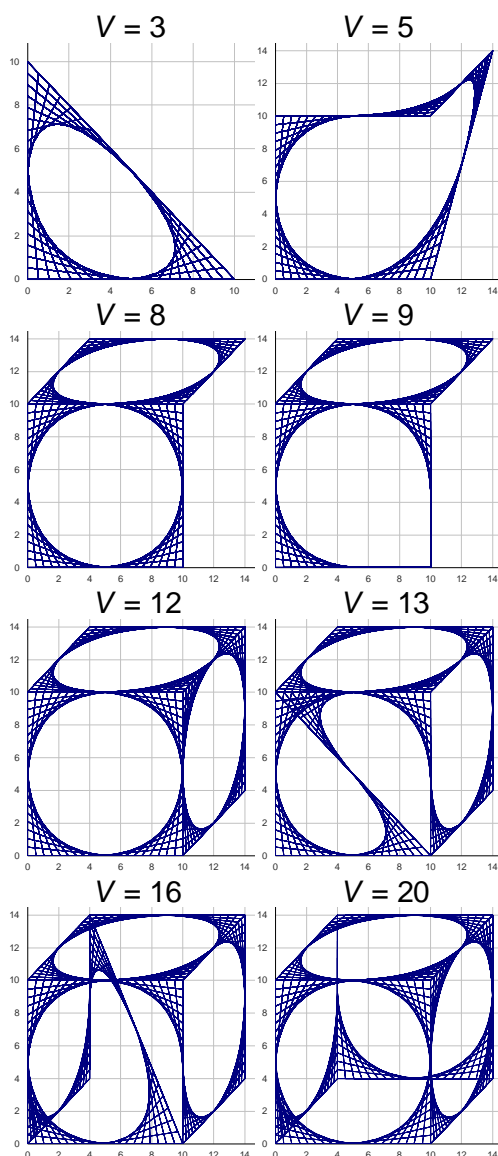
| Vertices | X  | Y  |
|----------|----|----|
| 1        | 10 | 0  |
| 2        | 0  | 0  |
| 3        | 0  | 10 |
| 4        | 10 | 10 |
| 5        | 14 | 14 |
| 6        | 4  | 14 |
| 7        | 0  | 10 |
| 8        | 10 | 10 |
| 9        | 10 | 0  |
| 10       | 14 | 4  |
| 11       | 14 | 14 |
| 12       | 10 | 10 |
| 13       | 0  | 10 |
| 14       | 0  | 0  |
| 15       | 4  | 4  |
| 16       | 4  | 14 |
| 17       | 4  | 4  |
| 18       | 14 | 4  |
| 19       | 14 | 14 |
| 20       | 10 | 10 |
| 999      |    |    |

**999 above means "vertex not used"**

*Table 1. Dashboard for creating images based on user-determined vertices.*

## 6.2 Creating your own Initials

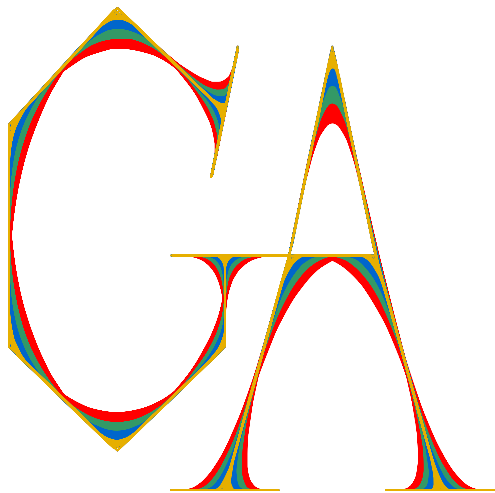
For students who are just learning about Cartesian coordinates, asking them to create their own initials in string art, provides an exercise that can spark excitement and creativity.



*Fig 13. Creating a cube based on Table 1. Images for various values of V.*

Section 19.6 of [1] provides suggestions for laying out initials. Among them, the most important are to start in the middle, use graph paper, and cell reference vertices that are used multiple times. The GA shown in Fig 14 used 33 vertices as

noted beneath the image. The image started at the connection between G and A and returned to that vertex four more times at vertices 19, 20, 28 and 33. The 19, 20 (after G was created) and 33, 1 pairs of vertices form zero jumps. This ensures that there are no curves connecting the G to the A (since  $S > P$ ). The Excel file used to create this image allows up to 40 vertices and those vertices need not coincide by color [1].



|    |    |    |                                    |      |   |  |
|----|----|----|------------------------------------|------|---|--|
| 33 | 47 | 34 | Red                                | 1551 | V; # of vertices used. $V < 41$             |  |
| 33 | 47 | 25 | Green                              | 1551 | S; # of subdivisions per side               |  |
| 33 | 47 | 17 | Blue                               | 1551 | P; # of subdivisions between Points         |  |
| 33 | 41 | 8  | Gold                               | 1353 | You can change the yellow and pink cells    |  |
| V  | S  | P  | Lines per color (should be < 2000) |      | You can link to gray cells to pink vertices |  |

Fig 14. An example of creating initials using ESA.

## 7. Summary

*Electronic String Art* combines art and mathematics in a way that allows users to learn mathematics more deeply, simply by adjusting parameters and watching what happens. ESA was created for independent explorations but it is useful in more formal K-12 classroom settings to support teaching of various topics in a visually appealing way.

## Notes

The links below take you directly to the web version [2] set to that figure. There are various *Drawing Modes*, start with *Fixed Count Line Drawing*, FCLD, and set *Drawn Lines*, DL, as noted then adjust drawing speed to your taste.

- The web version of [Fig 4](#), set DL = 2.
- The web version of [Fig 5](#), set DL = 11.
- The web version of [Fig 6](#), set DL = 12.
- The web version of [Fig 7](#), set DL = 7.
- The web version of [Fig 8](#), set DL = 3.
- The web version of [Fig 10](#), set DL = 3.
- The web version of [Fig 11](#), set DL = 3.
- The web version of [Fig 12](#), set DL = 2.

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