FROM LINES TO CIRCLES: RETHINKING DESIGN COORDINATES
Paper

Topic: Design AND Architecture

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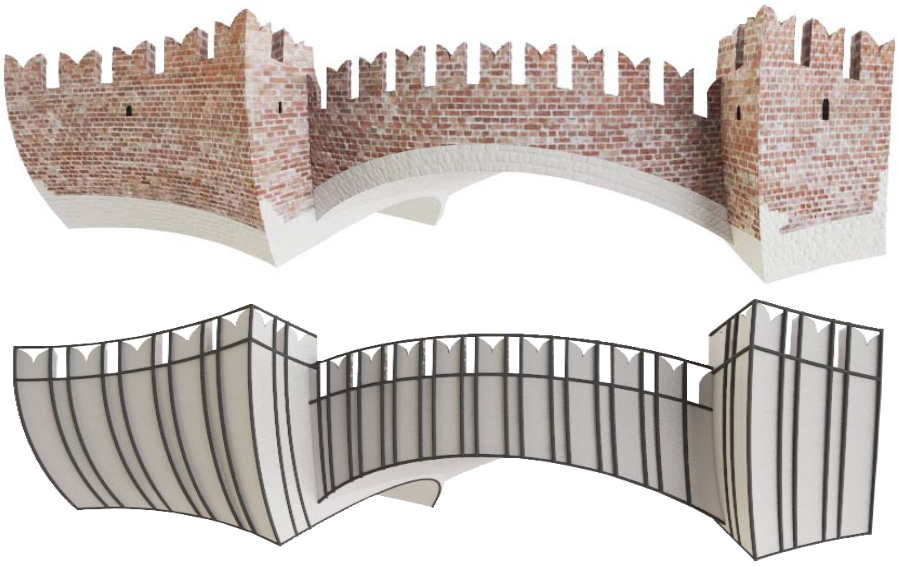
Abstract
"Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show." Bertrand Russel

Typical design thinking takes place within a rectangular coordinate system of non-curved three-dimensional Euclidean space. The usual coordinates, Cartesian coordinates, provide a natural way of segmenting space into rectangular prisms, and these are then used as a basis for thinking about, and constructing shapes within, that space. This approach, however, is pretty rigid, given that there are many more interesting, curved ways of segmenting space. Here we will propose a paradigm to break out of that restriction by using curved space, but while still using a rectangular coordinate system. Inspired by [2], we will describe coordinate systems in which straight lines become arcs of circles, and thusly, rectangular prisms become ones with arcs of circles for edges, and with faces that are swept out by circles, as parts of Dupin cyclides. We will discuss how one could use these systems of coordinates to re-think design in three-dimensional space.



Generation of part of the *Ponte di Castelvecchio* in Verona in curved space; the lower image shows, in black, the "grid lines"

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Key words: curved space, coordinate systems, Dupin cyclides

Main References:

[1] Bobenko, A.I. & Huhnen-Venedey, E. Geom Dedicata (2012) 159: 207. https://doi.org/10.1007/s10711-011-9653-5

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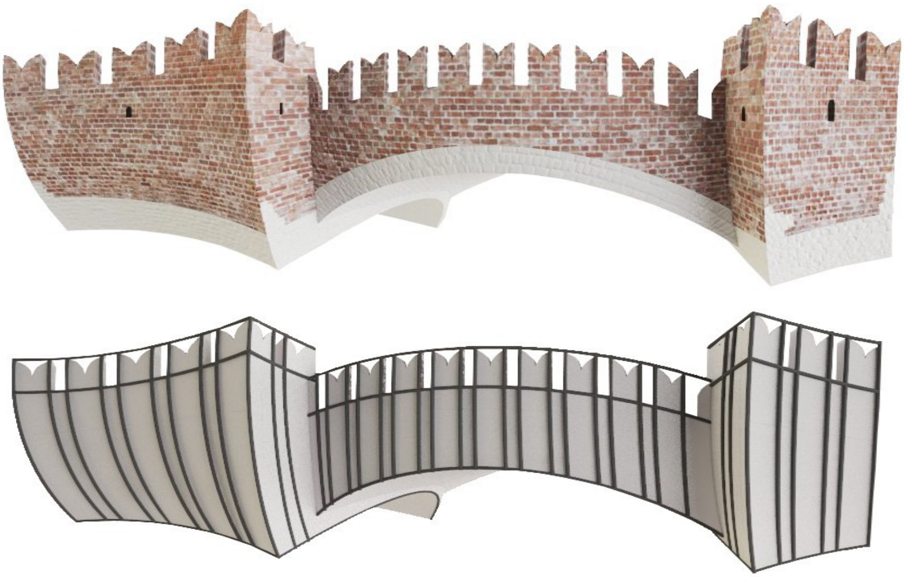

Generation of part of the *Ponte di Castelvecchio* in Verona in curved space; the lower image shows, in black, the "grid lines", which are arcs of circles, that were used for the basis of the design.

Key words: curved space, coordinate systems, Dupin cyclides

0. Philosophical interpretation

Since ancient times, humans have sought to find ways to discern the spatiality of the real world, and the best way to handle interacting with, and designing within, it. With the help of the tools of perception, like perspective, they have directed with the laws of Nature to create and develop. In the case of perspective, this means finding a meaningful way to represent of objects from a specific viewpoint on a two- dimensional plane, using the key notion of relative dimension to convey depth. At the heart of this, is the natural, rectilinear use of rectangular (or Euclidean) coordinates.

And when it comes to creating, and handling, three-dimensional objects, people have gleaned much with this usual rectilinear approach. However, it is a system that is quite rigid, in that the main building components are non-curved – for example lines, planes, etc. While these are, in many cases, the most-graspable, and easy-to- draw, there is much more that three-dimensional space can be: namely, it can be *curved*.

To step in this direction, we consider the natural generalization, Möbius Geometry, where one uses circles, instead of lines, and spheres, instead of planes, etc.[3] By way of realization, and implementation, we do this motivated by the paper by Bobenko and Huhnen-Venedey [2], in which they formulate how one could segment three-dimensional space into specially-curved hexahedrons instead of regular cubes. For a comparison of these two, see Figure 0. The further novelty of this approach is that we developed a workflow to deform a rectilinear object, into a curved one, within this framework. The implementation, and example workflow, of this is discussed in Section 1, while the mathematical foundations of this are discussed in Section 2.

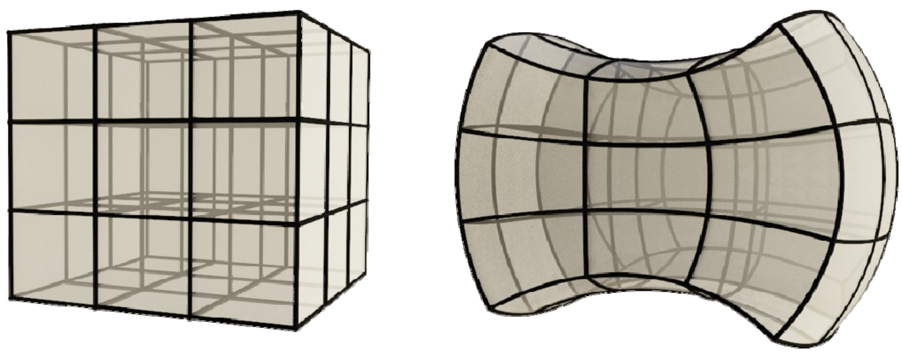

Figure 0: *Left* Cubes segmenting three-dimensional space. *Right* Specially-curved cubes segmenting three-dimensional space. (Made in Rhinoceros 5)

1. Motivation and Implementation

Motivated by the paper by Bobenko and Huhnen-Venedey [2], we have implemented a means of exploring and manipulating "discrete triply-orthogonal (coordinate) systems". These "systems" are formed of hexahedrons, in which the points of each face lie on a circle, and in our implementation, to reduce the number of degrees of freedom, we have used the further constraint that all points on diagonal also lie on a circle. Further discussion of this is in Section 2.

1.1 Application of Miquel's Theorem to Construct a Hexahedron

At the basis of our implementation, we use Miquel's Theorem in two dimensions. This theorem states that, given a triangle ABC with arbitrary points A', B', and C' on the respective sides BC, AC, and AB, the three circumcircles to the triangles $AB'C'$, $A'BC'$, and $A'B'C$ always intersect in a single point, called the *Miquel point*; see Figure 1 for a depiction of this.

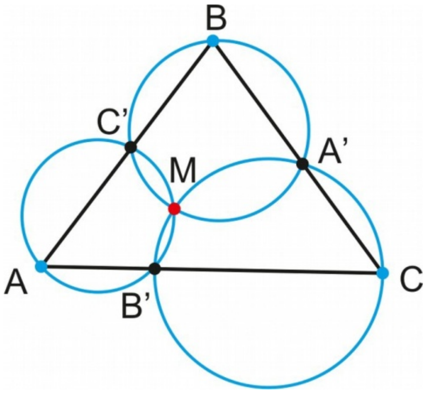

Figure 1: Miquel's Theorem in two dimensions. (Made in CorelDRAW x7)

In order to apply this theorem to a hexahedron with the aforementioned constraints, we first stereographically project the hexahedron to the plane; see Figure 2a. The result of this application of Miquel's theorem is summarized as follows, as shown in Figure 2a/b: given four blue points, and choosing the green point on the orange circle, the black points can be uniquely determined via Miquel's Theorem, and the red point is the Miquel point.

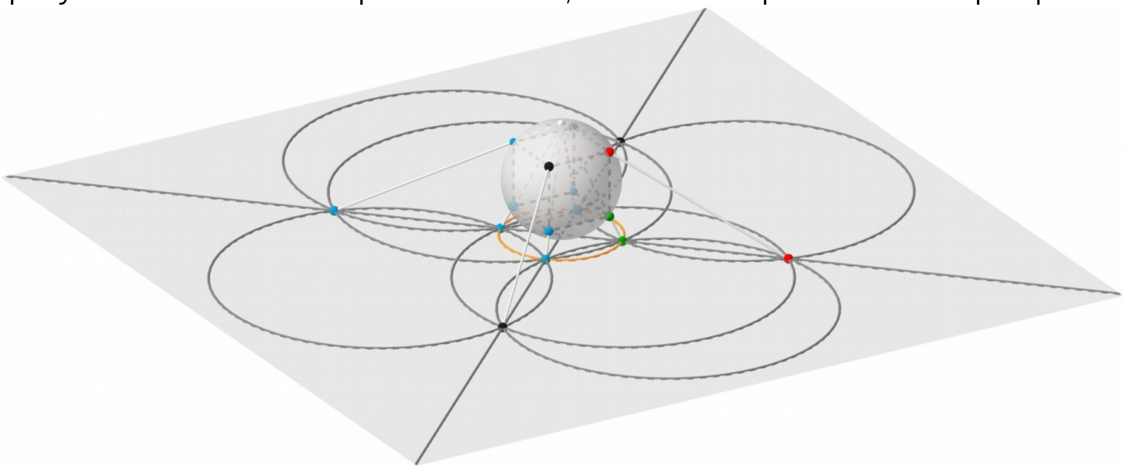

Figure 2a: Stereographic projection of a cube. (Made in GeoGebra 5)

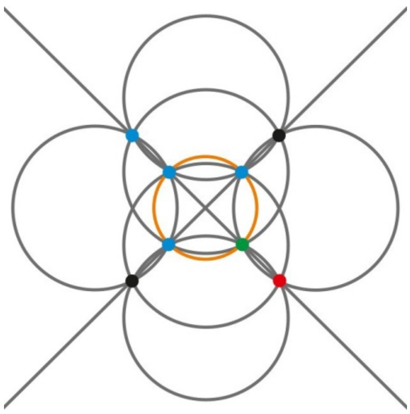

Figure 2b: Projection of the cube on the plane, with the circles from the three-dimensional version of Miquel's Theorem. (Made in CorelDRAW x7)

1.2 Discrete Triply-orthogonal System

We are going to focus on the lattices with an equal number of hexahedrons in their three spatial directions. For this type of lattices, the number of manipulable points is $3n+2$, with one point being chosen on a circle, where n is the number of hexahedrons in one direction – this is $9n+4$ degrees of freedom – shown in Figure 3, left. In this way, if we move any one of those points, the entire lattice will be modified so that the faces, and diagonals, lie on circles.

Once we have defined the lattice, we can pick a frame at one of its points, which we will use to create faces, which will be discussed more in Section 2. This frame is propagated across the mesh by defining it at adjacent points with a reflection across the bisecting plane. From these frames, circular arcs are determined between adjacent points, so that it is tangent to the frames at each point, as in Figure 3, right.

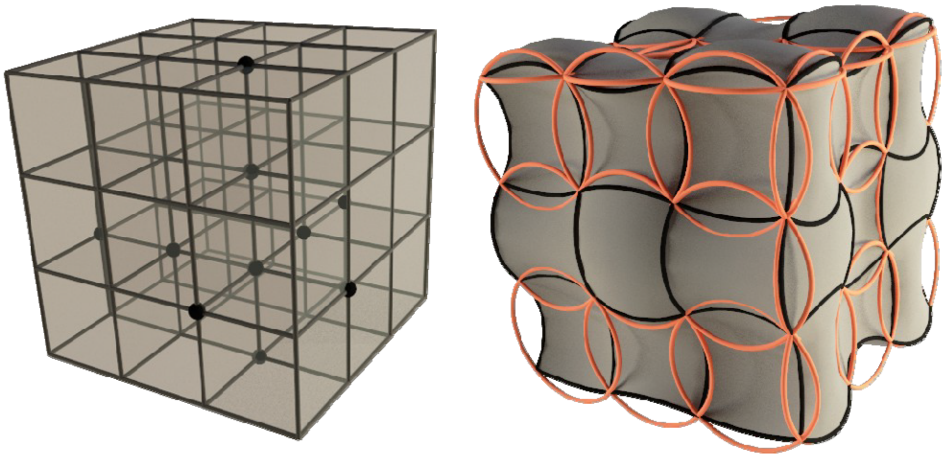

Figure 3: *Left* Manipulable points in the lattice. *Right* The resulting discrete triply-orthogonal system.
(Made in Rhinoceros 5)

1.3 Example

To better understand design within this framework, we will go over an example of how it can be used to create curved elements in three-dimensional space. We will break out of the restrictions of the traditional (non-curved, Euclidean) space by using the curved space from the discrete triply-orthogonal systems discussed before.

For the choice of object to be represented, we think that there is no better option than a part of the *Ponte di Castelvecchio* located in Verona, as in Figure 4a, top. First, we delineate the model into hexahedrons, as shown in Figure 4a, middle. Then, we choose a frame at one of the corners, to generate a curved version, as shown in Figure 4a, bottom. Lastly, we refine the model using the circular arcs from the faces to construct details, obtaining the final model, as shown in Figure 4b.

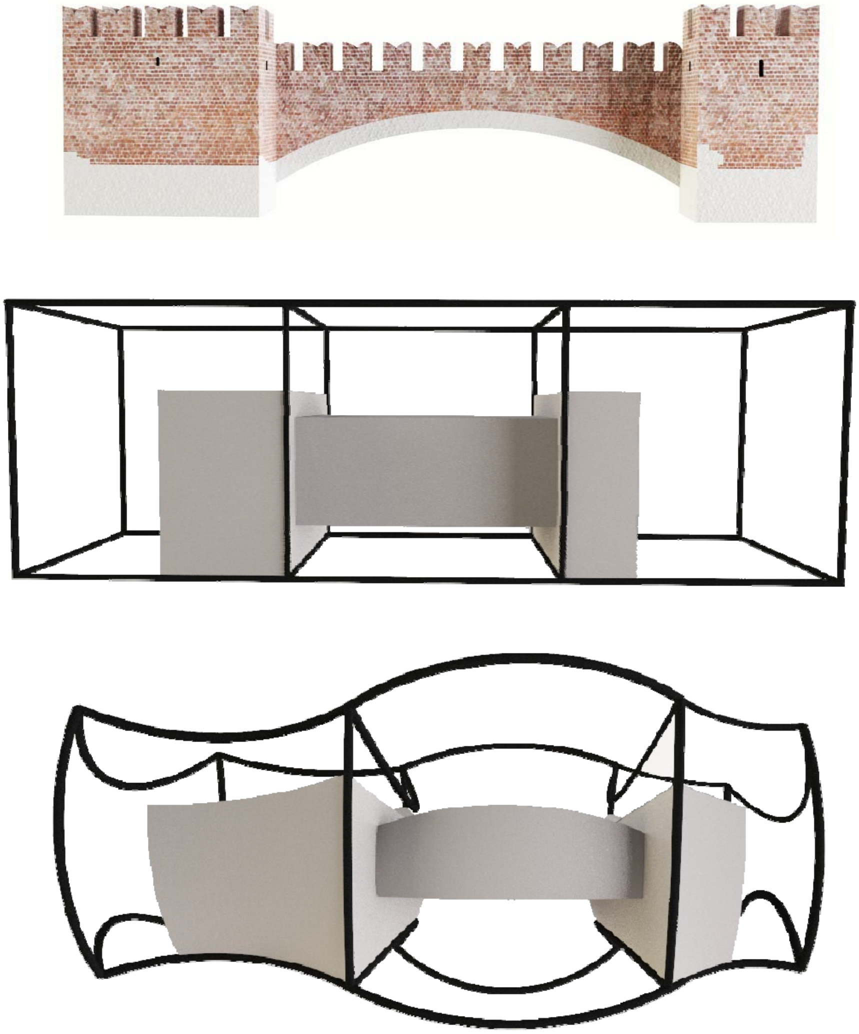

Figure 4a: Starting steps in workflow to generate a curved model: *Top* Starting model. *Middle* delineating the model into hexahedrons. *Bottom* Getting the curved model after a choice of a frame. (Made in Rhinoceros 5)

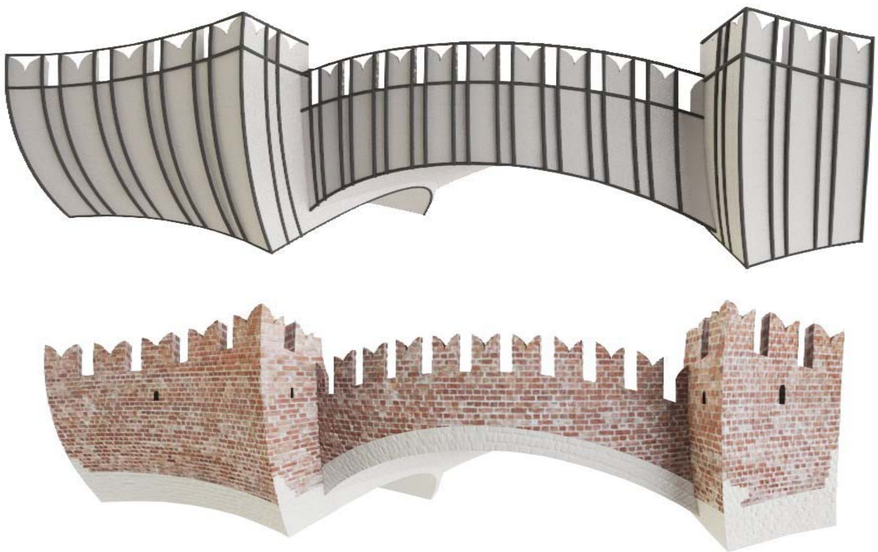

Figure 4b: Last steps in workflow to generate a curved model: *Top* Refined curved model using circular arcs. *Bottom* Final model. (Made in Rhinoceros 5)

2 Mathematical Structure

In this section, we will go over the mathematical underpinnings of our article. The majority of it comes from the paper by Bobenko and Huhnen-Venedey [2], which formulates a notion of a "discrete triply-orthogonal system".

2.1 Smooth Case

A "triply-orthogonal system" is best described with a familiar example, Figure 5: the usual way of thinking about three-dimensional space, in which, at every point, there is a natural notion of the x-, y-, and z-directions. These directions are, at each point, all orthogonal to each other, which is to say, they are *triply-orthogonal*. Moreover, these directions are consistently parallel: all the x-directions are parallel to each other, and so are the y- and z-directions, respectively. The collection of these directions at each point is called a *frame*.

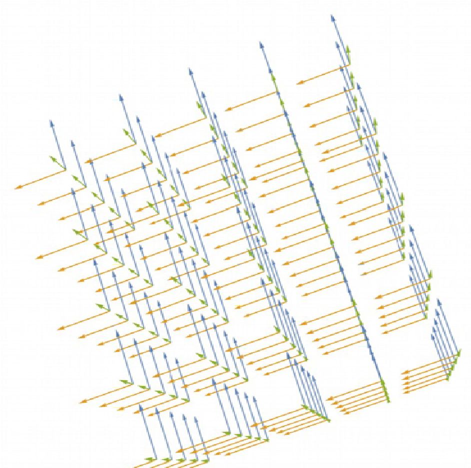

Figure 5: The usual triply-orthogonal system used in three-dimensional space, shown as a collection of frames. (Made in Mathematica 11)

To generalize this, one could think about what would happen if the choice of frames at each point was allowed to vary, keeping the triple-orthogonality but relaxing the parallelity, as in Figure 6. In this way, one would obtain generally a triply-orthogonal system.

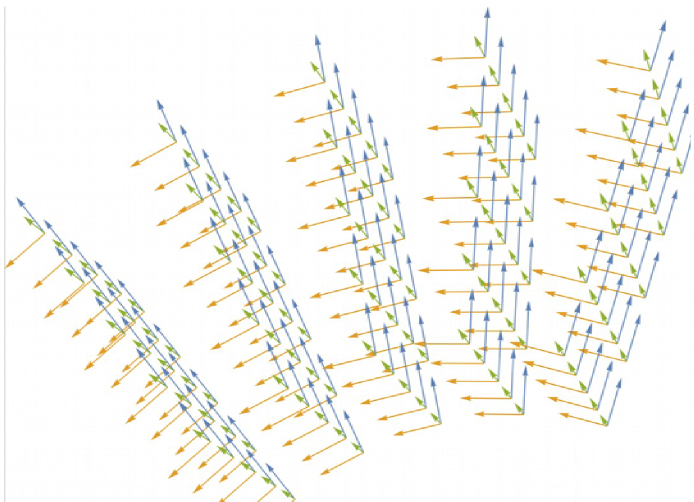

Figure 6: Another example of a triply-orthogonal system in three-dimensional space, shown as a collection of frames. (Made in Mathematica 11)

2.2 Discretization

To make a triply-orthogonal system more tangible and more-easily implementable, it is possible to discretize the notion, as is done in a paper by Bobenko and Huhnen- Venedey, [2]. Building off well-established discretization theory for surfaces, they call a lattice of points a *circular net* [2, Definition 3.1], which is a *discretized* triply- orthogonal system, if there is the constraint that all corresponding sets of points lie on circles, as depicted in Figure 7. With that constraint, there are many examples in which the points are not regularly-spaced, as in Figure 8.

Figure 7: A circular net, which has regularly-spaced points: there are circles through sets of points corresponding to each face of each cube. (Made in GeoGebra 5)

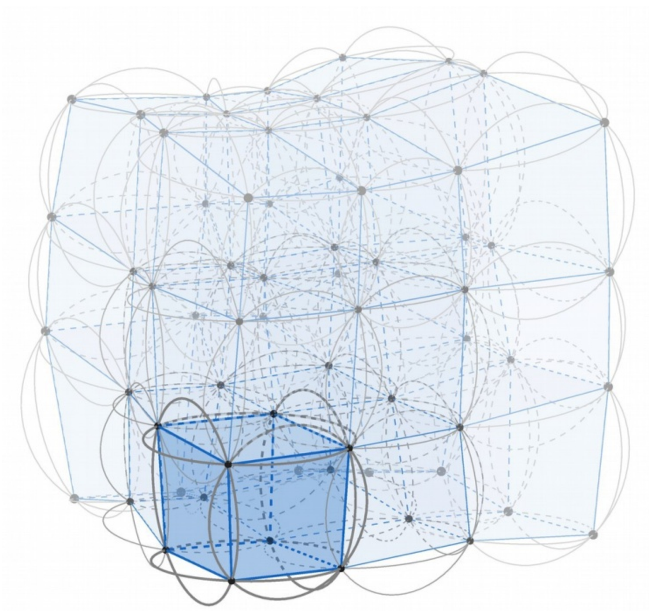

Figure 8: A circular net, which does not have regularly-spaced points. (Made in GeoGebra 5)

To allow for more deformation, Bobenko and Huhnen-Venedey add in a choice of a frame for any single point of the lattice; this frame is then propagated across the lattice to form a *cyclidic net*.[2, Definition 3.3] This frame enables sections of special surfaces, called “Dupin cyclides”, to form the faces, instead of simple planar ones, which gives rise to the qualifier *cyclidic* in their name. These surfaces, *Dupin cyclides*, are surfaces characterized by being made up of two perpendicular families of circles, in a special[1] way, which is that they follow directions where the surface curves most. An example of a Dupin cyclide is in Figure 9, along with a highlighted section, which could be a face in a cyclidic net. And, an example of a cyclidic net is in Figure 10. In this way, they have described a way of breaking up three-dimensional space into *cyclidic hexahedron* pieces, determined by a lattice and a frame.

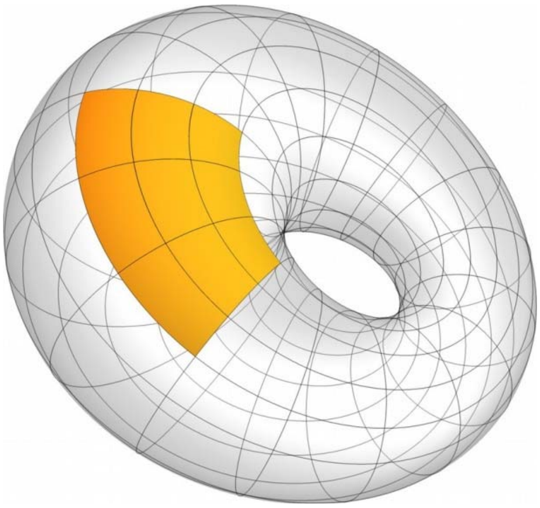

Figure 9: A Dupin cyclide, shown with its two special families of circles, and a highlighted section. (Made in Mathematica 11)

1 These two families of circles are the principal curvature lines of the surface.

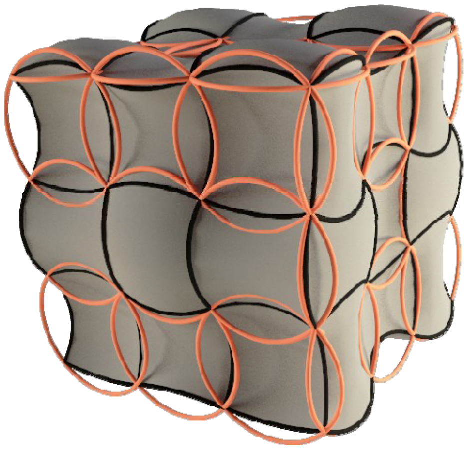

Figure 10: A cyclidic net, shown with orange circles, which constrain the lattice, and with black circular-arcs, which are edges of the cyclidic hexahedrons. (Made in Rhinoceros 5)

The natural next question is: how does one think about the inside of those cyclidic hexahedrons, so that it is consistent with this discretized triply-orthogonal system, of which it is part? Luckily, this question is answered by Bobenko and Huhnen- Venedey: they show that it is possible to parametrize the inside of each one of those cyclidic heaxhedrons with their own three families of triply-orthogonal Dupin cyclides. [2, Theorem 3.9, Corollaries 3.10 & 3.11] This is done in such a way that the intersection of any the Dupin cyclides in any of those families with the faces of the cyclidic hexahedron, is part of a circle from the families of circles that make each of them up. See Figure 11 for a depiction of that.

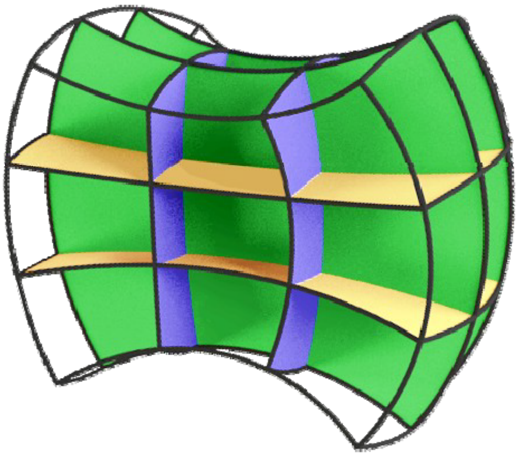

Figure 11: Parametrization of the inside of a cyclidic hexahedron, with two Dupin cyclides shown from each three families, distinguished by colors. (Made in Rhinoceros 5)

3.3 Remark on the Implementation

In order to simplify implementation, more circles were used to constrain the lattice, thereby reducing the degrees of freedom; see Figure 12.

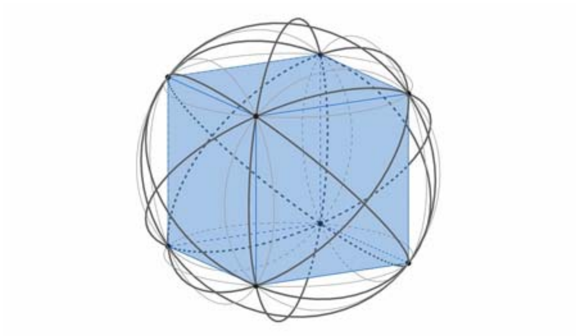

Figure 12: These are circles constraining the lattice in the implementation, where the circles in a darker color are the circles not present in the constraints of a generic circular/cyclidic net. (Made in GeoGebra 5)

With these further circles and a choice of a frame at one of the points, there is a further implication that the resulting cyclidic hexahedron is a Möbius transformation of a regular cube; this is to say that the cyclidic hexahedron can be gotten from a regular cube after a series of translations, scalings, rotations, reflections, and inversions about spheres.[1]

Acknowledgements

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