Approximation Theory

Felipe Cucker and Héctor Rodríguez
Dept. of Mathematics and School of Creative Media
City University of Hong Kong
HONG KONG
e-mail: macucker@cityu.edu.hk, smhect@cityu.edu.hk

Abstract. Approximation Theory is an art-research project in visual mathematics and data aesthetics. It consists of a series of prints and videos that visualize the mathematical idea of approximation.

1 Description

Approximation Theory is an art-research project in visual mathematics and data aesthetics. It consists of a series of prints and videos that visualize the mathematical idea of approximation.

The methodology used in the work involves the choice of a set of fixed dictionaries or databases of images. Each dictionary has its own distinctive quality. For instance, some consist of linear or curved elements generated procedurally.

Figure 1 A few images from the rectilinear dictionary

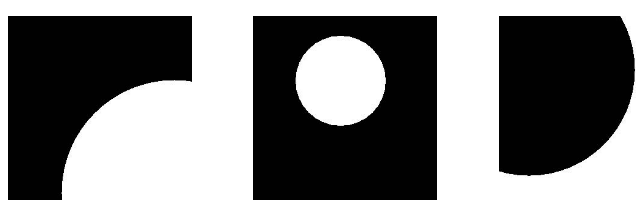

Figure 2 A few images from the curvilinear dictionary

Other dictionaries might consist of more complicated images, such as frames from movies or Chinese characters.

Figure 3 A few images from the Chinese dictionary

Any other image can then be reconstructed as a weighted superposition of all or some of the images in the dictionary. For instance, this source image from the 1936 Summer Olympics can be reconstructed by superposing a selection of 3840 images (each of them of the "linear" dictionary).

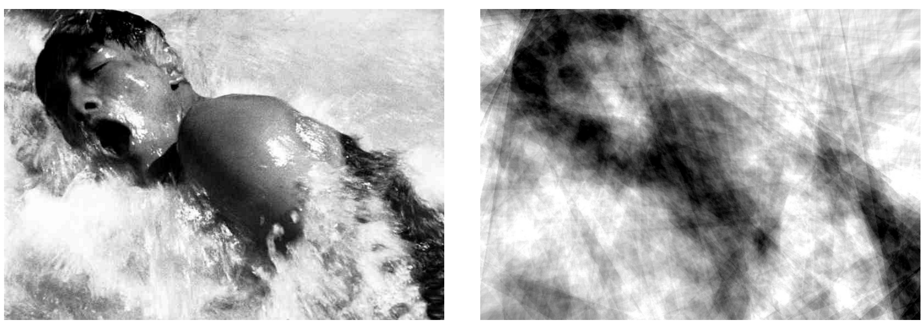

Figure 4 An image of a swimmer and a reconstruction of it

There exist mathematical procedures that identify how to superpose a given subset of images from the dictionary in a manner that most closely reconstructs or "approximates" the source image.

The character of the approximation depends on two kinds of factors: qualitative and quantitative.

The qualitative aspect has to do with the character of images in the dictionary, for instance whether they are linear or curved. Figure 5 shows an image and three reconstructions obtained using the three dictionaries shown in Figures 1–3. While the three results resemble the original, each has its own distinctive personality.

The quantitative aspect has to do with the number of images from the dictionary that are used in the reconstruction. The sequence of digital prints in Figure 6 shows several reconstructions of one source image using progressively larger selections from the same dictionary.

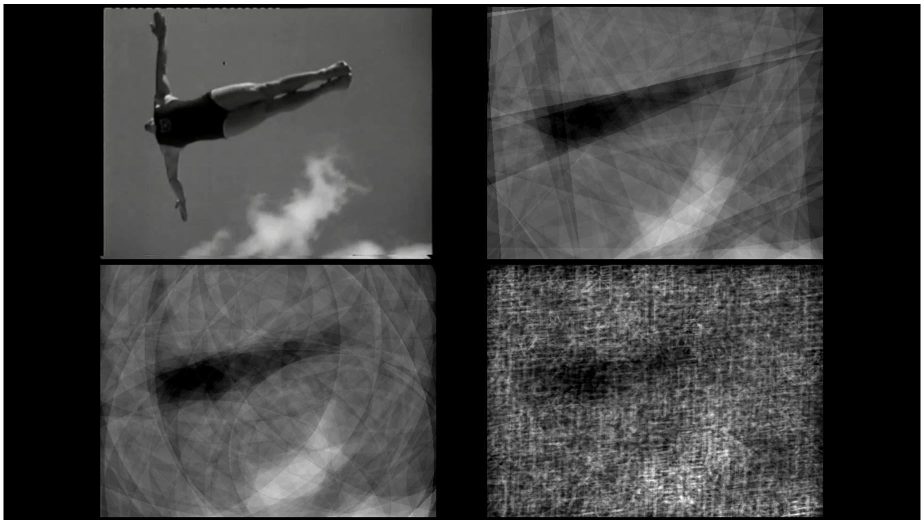

Figure 5 Three reconstructions with different dictionaries

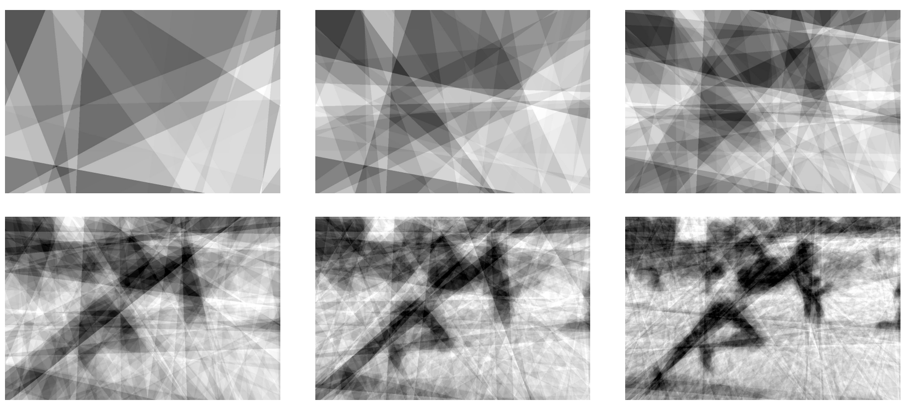

Figure 6 Reconstructions using 30, 60, 120, 360, 720, and 1920 images from the dictionary

Our work uses source images drawn from the 1936 Summer Olympics, as shot by director Leni Riefenstahl. These images express a fascination with the human form, and one of our main concerns has to to do with the representation of the body in an algorithmic age. The images are also drawn from an age where the threat of fascism was very real, and manifested itself in the cult of the body. We are also confronting a growing sense of populist authoritarianism around the world. Our response to this threat is to reassert the value of rational analysis and the integration of art, science, and cultural critique. An additional reason to choose Riefenstahl's movie is that it naturally provides a diverse, yet thematically coherent, collection of short clips, rich in movement. The reconstruction of these clips with different dictionaries allows for a visual comparison of the dictionaries' characters.

This project can be exhibited as a set of prints that explore different aspects (both qualitative and quantitative) of mathematical approximation theory, together with videos that compare different reconstructions. The number of prints and videos can be adjusted to take into account venue and equipment constraints.

2 Mathematical Approximation

This section contains a more detailed description of mathematical approximation theory, and can be skipped.

The idea of approximation underpins many aspects of digital technology, including machine learning, computer vision, and digital image and sound processing. In many of these applications, it is vital to replace a complicated function (for instance, one that is expensive to compute or tedious to describe) by a simpler one. Digital data compression, for instance, often involves the replacement of image or audio data by a shorter description. Approximation theory aims to find the best approximation of the desired input.

The context of approximation theory is the following: there is a target function f which we want to approximate with another function g to be selected from within a set S of approximant functions. The main problem is that of estimating how good this g can be, that is, how small can the error of replacing f by g be. This problem presupposes a way of measuring this error (usually a distance in a space of functions where both f and g live). In addition, it depends on both the target function f and the set of approximant functions S. This dependence has both quantitative and qualitative aspects.

On the quantitative side there are parameters, such as the dimension of the linear space spanned by S (typically, when S spans a finite dimensional linear space) or the radius of S (typically, when S is a closed ball in a linear space, possibly of infinite dimension).

On the qualitative side there are factors such as the kind of functions in S (e.g., polynomials, radial-basis functions, splines, etc.) and the quality of the function f itself (how smooth it is, how rapidly it oscillates, etc.).

Our project is an experimental exploration of the quantitative and qualitative aspects of approximation.

Intuitively, a digital image can be viewed as a position or a point in a space of possible images. More precisely, an image of N (grayscale or "black-and-white") pixels is identified with a vector of N real numbers in the interval $[0, 1]$, with 0 being black and 1 being white. Our target function (for instance a photograph that we wish to reconstruct) is a point in this N-dimensional vector space. A set S of approximant functions is a subset of the unit cube in this space. In other words, S is a set of N-pixel images (the "dictionary").

Any target image is to be approximately reconstructed as a weighted sum (linear combination) of a selection of images from the dictionary. Intuitively, we can think of this procedure as a superposition or mix of brightened or darkened versions of the various images in the dictionary. By changing the intensities of the images in the dictionary in appropriate ways, the resulting combination will resemble approximately the target image. The "appropriate ways" of mixing the dictionary are determined automatically by mathematically projecting the target image onto the subspace spanned by the images in the dictionary.

The quantitative side mainly involves the number of images to be used in any approximation, i.e., the size of the dictionary. We might choose to employ only a relatively small subset of images in the dictionary. As the number of images employed decreases, the fidelity of the resulting reconstruction also decreases, and vice versa.

The qualitative side involves the kinds of images to be included in the dictionary. Our project explores how the use of different kinds of images (for instance geometrically abstract frames consisting of lines or checkerboards, photographs of different locations in the world or still frames from classical films) affects the visual quality of the reconstruction.

We are currently exploring the use of advanced mathematical tehcniques, such as ℓ_1-minimization, affect the visual character of the result.